THE $O(n^2)$ ALGORITHM FOR THE EIGENPROBLEM OF AN ε-TRIANGULAR TOEPLITZ MATRICES IN MAX-PLUS ALGEBRA

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ABSTRACT

For a general $n \times n$ real matrix $A = (a_{ij})$, there exist standard $O(n^3)$ algorithms to find λ, $x_1, \dots, x_n$ such as

$$\max_{j=1,2,\dots,n}(a_{ij} + x_j) = \lambda + x_i \quad \text{for all } i = 1,2,\dots,n.$$

It is known that λ is unique and equals to the maximum cycle mean of $A = (a_{ij})$. Paper considers the case when $A = (a_{ij})$ is an ε-triangular Toeplitz matrix, i.e. a diagonal-constant matrix with $a_{ij} \in R \cup \{-\infty\}$, $a_{ij} = \varepsilon \le -(n-1)\max_{i \le j}|a_{ij}|$ for $i > j$, $a_{ij} \in R$ for $i \le j$, and present algorithms to determine λ, $x_1, \dots, x_n$ in $O(n^2)$ time.

Keywords: *extremal eigenvalue, eigenvector, Toeplitz matrix*

1. INTRODUCTION

Problems in many research areas, such as system theory, graph theory, scheduling, knowledge engineering, transfer data, can be formulated in the language of extremal algebras, in which the addition and multiplication of vectors and matrices are formally replaced by operations of maximum and plus, respectively. The steady states of discrete events processes correspond to eigenvectors of max-plus matrices, see [5], hence the characterization of the extremal eigenspace is important for the applications. In some cases, the investigation is more efficient, if the considered matrix has special properties. Many efficient solutions of problems concerning special classes of matrices were described in [1, 2, 8, 9, 16–18]. Problems related to eigenvalue and eigenvectors of special case of Toeplitz matrices in a max-plus algebra were studied in papers [17, 18] in which efficient algorithms for various special cases were presented. In general, the problem of computing the eigenvalue and eigenvectors can be solved in $O(n^3)$ time. In this paper, efficient techniques will be used for computing the metric matrix connected with the eigenspace of a given triangular Toeplitz matrix. The eigenspace and the eigenspace dimension will be completely described by this computation, which can be performed in $O(n^2)$ time.

By a max-plus algebra we understand the algebraic structure $(R^*, \max, +)$, where R^* is the set of all real numbers R extended by an infinite element $-\infty$ and $\oplus$, $\otimes$ are the binary operations on R^*: $\oplus = \max$ and $\otimes = +$. The infinite element is neutral with respect to the maximum operation and absorbing with respect to addition.

For any natural $n > 0$, we denote $N = \{1, \dots, n\}$. Further, R^*_n denotes the set of all $n \times n$ matrices over R^*. The operation $\otimes$ for matrices denotes the formal matrix product with operations $\oplus = \max$ and $\otimes = +$ replacing the usual operations $+$, $\times$, while the operation $\oplus$ for matrices is performed componentwise.

The problem of finding a vector $x \ne (-\infty, \dots, -\infty)^T$ and a value $\lambda \in R$ satisfying $A \otimes x = \lambda \otimes x$ is called the eigenproblem corresponding to the matrix A, the value λ is called *eigenvalue*, and x is called *eigenvector* of A.

Let $A = (a_{ij}) \in R^*_n$. The associated digraph D_A of the matrix A is defined as a couple $D_A = (N, \{(i,j) \in N \times N,\ a_{ij} > -\infty\})$. As usual a digraph D is called *strongly connected* if there is a directed path between any pair of nodes in D. The matrix A is called as *irreducible* if D_A is strongly connected, otherwise *reducible*. If p is a path or a cycle in D_A, of length $r = |p|$, then the weight $w(p)$ is defined as the sum of all weights of the arcs in p. If $r > 0$, then the mean weight of p is defined as $w(p)/r$. Of all mean weights of cycles in D_A, the maximal one is denoted by $\lambda(A)$.

We also denote $E(A) = \{i \in N; \exists\sigma = (i : i_1, \dots, i_k) : w(\sigma)/k = \lambda(A)\}$. The elements of $E(A)$ are called eigennodes (of A). A cycle σ is called *optimal* if $w(\sigma)/k = \lambda(A)$.

Full solution of the eigenproblem in the case of irreducible matrices has been presented in [6] and [11]. The problem of finding the eigenvalue $\lambda(A)$ has been studied by a number of authors and several algorithms are known for solving this problem. The algorithm described by Karp in [13] has the worst-case performance $O(n^3)$ and Howard's algorithm [4] of unproven computational complexity shows excellent algorithmic performance.

For $B \in R^*_n$ we denote, by $\Delta(B)$ the matrix $B \oplus B^2 \cdots \oplus B^n$, where B^s stands for the $(s-1)$-fold iterated product $B \otimes \cdots \otimes B$. Further, we denote $A_\lambda = \lambda^{-1}(A) \otimes A$ (here we have a formal product of a scalar value $\lambda(A)$ and a matrix A, i.e. $[A_\lambda]_{ij} = -\lambda(A) + a_{ij}$ for any $(i,j) \in N \times N$). It is shown in [6] that the matrix $\Delta(A_\lambda) = (\xi_1, \dots, \xi_n)$ contains at least one column ξ_i, the diagonal element of which is 0 and every such a column is the eigenvector (so called: fundamental eigenvector) of the matrix A. Moreover, every eigenvector of the A can be expressed as a linear combination of fundamental eigenvectors. The set of all fundamental eigenvectors will be denoted by F_A and its cardinality is denoted by $q = |F_A|$. We can say that $x, y \in F_A$ are equivalent if $x = \alpha \otimes y$ for some $\alpha \in R$. $Sp(A)$ denotes the set of all eigenvectors of a matrix A, called *eigenspace* of A.

From the definition of equivalent fundamental eigenvectors it follows that the set F_A can be replaced by any maximal set F'_A of fundamental eigenvectors such that no two of them are equivalent. Every such set F'_A will be called a complete set of generators (of the eigenspace). The cardinality of F'_A is called the dimension of $Sp(A)$, i.e. $|F'_A| = \dim(Sp(A))$.

Theorem 1.1 [6] *If $A \in R^*_n$ is an irreducible matrix then A has a unique eigenvalue equal to $\lambda(A)$, all eigenvectors of A are finite and the set of all eigenvectors is*

$$\left\{ \sum_{i \in E(A)}^{\oplus} \alpha_i \otimes \xi_i;\ \alpha_i \in R \right\}.$$

Theorem 1.2 [6] *Let $\xi_1, \ldots, \xi_n$ denote the columns of the matrix $\Delta(A_\lambda)$. Then*

(i) $j \in E_A$ if and only if $\xi_j \in F_A$

(ii) ξ_i, ξ_j are equivalent members of F_A if and only if the eigennodes i, j are contained in a common optimal cycle.

Let $\Delta(A_\lambda) = (\xi_{ij})$. It follows from the definition of $\Delta(A_\lambda)$ that ξ_{ij} is the weight of a heaviest path from i to j in D_A. Hence, $\Delta(A_\lambda)$ can be computed in $O(n^3)$ operations using the Floyd-Warshall algorithm [14]. By trivial search and comparisons one can then find a complete set of fundamental eigenvectors among the columns of $\Delta(A_\lambda)$, using at most $O(n^3)$ operations.

A general spectral theorem for reducible matrices was presented in [3] and [10]. Note that the set $Sp(A)$ is in general not a max-plus subspace for reducible $n \times n$ matrices which may have up to n eigenvalues [10].

It is known from [3] that every matrix $A = (a_{ij}) \in R^*_n$ can be transformed to a Frobenius normal form

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & A_{13} & \ldots & A_{1,r-1} & A_{1r} \\ -\infty & A_{22} & A_{23} & \ldots & A_{2,r-1} & A_{2r} \\ -\infty & -\infty & A_{33} & \ldots & A_{3,r-1} & A_{3r} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\infty & -\infty & -\infty & \ldots & A_{r-1,r-1} & A_{r-1,r} \\ -\infty & -\infty & -\infty & \ldots & -\infty & A_{rr} \end{pmatrix}.$$

where $A_{11}, \ldots, A_{rr}$ are irreducible square submatrices of A. If A is in a Frobenius normal form then the corresponding partition of the node set N of D_A will be denoted as $N_1, \ldots, N_r$. It follows that each of the induced subgraphs $D_A[Ni]$ $(i = 1, \ldots, r)$ is strongly connected and an arc from N_i to N_j in D_A exists only if $i \leq j$. The symbol $N_i \to N_j$ means that there is a directed path from a node in N_i to a node in N_j in D_A.

Theorem 1.3 [3] *(Spectral theorem)*

Let $A = (a_{ij}) \in R^*_n$ be a matrix in the Frobenius normal form and $\lambda(A_{ii})$ be the (unique) eigenvalues of the diagonal blocs A_{ii}, $i \in \{1, \ldots, r\}$. Then $\lambda(A)$ is an eigenvalue of A if and only if there is an index j such that $\lambda(A) = \lambda(A_{jj})$ and $\lambda(A_{jj}) \geq \lambda(A_{ii})$ for every i for which $N_i \to N_j$.

Corollary 1.1 [3] *Every $n \times n$ matrix has at most n eigenvalues and the biggest one is $\lambda(A)$.*

Corollary 1.2 [3] *Let $A = (a_{ij}) \in R^*_n$ be a matrix in the Frobenius normal form. Then $\lambda(A) = \max\limits_{1 \leq i \leq r} \lambda(A_{ii})$.*

The aim of this paper is to show that in special case, when the matrix A is ε-triangular Toeplitz, the computations can be performed in a more efficient way as Karp algorithm works.

2. EIGENVALUE OF A TOEPLITZ MATRIX

A matrix $A = (a_{ij}) \in R^*_n$ is a Toeplitz matrix generated by the sequence b over R^*, $b = (b_{-n+1}, \ldots, b_{-1}, b_0, b_1, \ldots, b_{n-1})$, if $a_{ij} = b_{j-i}$ holds for all $i, j \in N$, i.e. the matrix A takes the form

$$\mathbf{A} = \begin{pmatrix} b_0 & b_1 & b_2 & \ldots & b_{n-2} & b_{n-1} \\ b_{-1} & b_0 & b_1 & \ldots & b_{n-3} & b_{n-2} \\ b_{-2} & b_{-1} & b_0 & \ldots & b_{n-4} & b_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ b_{-n+2} & b_{-n+3} & b_{-n+4} & \ldots & b_0 & b_1 \\ b_{-n+1} & b_{-n+2} & b_{-n+3} & \ldots & b_{-1} & b_0 \end{pmatrix}$$

A Toeplitz matrix is known if its first row and first column are given.

Theorem 2.1 *Let $A = (a_{ij}) \in R_n$ be a Toeplitz matrix. Then there exists an optimal cycle containing node 1 $(1 \in E(A))$.*

Proof. Suppose $A = (a_{ij}) \in R_n$ is a given Toeplitz matrix and $\lambda(A) = \frac{a_{i_1 i_2} + \cdots + a_{i_{k-1} i_k}}{k}$, $\sigma = (i_1, \ldots, i_k)$ and $\min\{i_1, \ldots, i_k\} = i_1 > 1$. Let $i_1 - 1 = r$, then for the elements a_{ij} of Toeplitz matrix A is fulfilled

$$a_{ij} = b_{(i-j)} = b_{(i-r-(j-r))} = a_{i-r, j-r}$$

hence the cycle σ there exists a cycle $\sigma' = (1, i_2 - r, \ldots, i_k - r)$ with the property

$$\lambda(A) = \frac{a_{1, i_2 - r} + \cdots + a_{i_k - r, 1}}{k}.$$

□

Since $\lambda(\alpha + \omega A) = \alpha + \lambda(A)$ for $\alpha \in R$ (see [6]) we shall assume w.l.o.g. that diagonal entries of A are equal to 0.

A Toeplitz matrix $A = (a_{ij}) \in R^*_n$ generated by the sequence b over R^*, $b = (b_{-n+1}, \ldots, b_{-1}, 0, b_1, \ldots, b_{n-1})$ is called ε-triangular Toeplitz matrix if $b_i \in R$ for $i \in \{1, \ldots, n\}$ and $-\infty \leq b_i = \varepsilon \leq -(n-1) \max\limits_{j \in \{1, \ldots, n\}} |b_j|$ for $i \in \{-n+1, \ldots, -1\}$, i.e.

$$\mathbf{A} = \begin{pmatrix} 0 & b_1 & b_2 & \ldots & b_{n-2} & b_{n-1} \\ \varepsilon & 0 & b_1 & \ldots & b_{n-3} & b_{n-2} \\ \varepsilon & \varepsilon & 0 & \ldots & b_{n-4} & b_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \varepsilon & \varepsilon & \varepsilon & \ldots & 0 & b_1 \\ \varepsilon & \varepsilon & \varepsilon & \ldots & \varepsilon & 0 \end{pmatrix}.$$

Denote the set of $n \times n$ ε-triangular Toeplitz matrices over R^* by $T_n(\varepsilon)$.

In the following two sections we will analyze the eigenproblem for the reducible and irreducible ε-triangular Toeplitz matrices.

Note that every ε-triangular Toeplitz matrix A is regular ([12]) unless $b_i = infty$ for $i \in N$, hence $\lambda(A)$ is the unique eigenvalue of A.

3. REDUCIBLE CASE: EIGENPROBLEM OF AN ε-TRIANGULAR TOEPLITZ MATRIX

For $A = (a_{ij}) \in T_n(-\infty)$ the symbol $D_A^{ac} = (N, E)$ denotes the digraph with node set N and $(i, j) \in E$ if and only if $i < j$. Evidently D_A^{ac} is a subgraph of D_A and it does not contain loops. One can easily see that a digraph D_A^{ac} is acyclic. Based on the fact that every matrix $A = (a_{ij}) \in T_n(-\infty)$ is in a Frobenius normal form we can apply the *Corollary 1.2* and formulate the following assertions.

Theorem 3.1 *Let $A = (a_{ij}) \in T_n(-\infty)$. Then $\lambda(A) = b_0 = 0$.*

Proof. The assertion follows from the fact that the only cycles which contain no lower diagonal element are $\pi = (i)$ which correspond to loops in D_A.
□

From the *Theorem 2.1* we know that 1 is an eigennode. It is well known [6] that to find an eigenvector it is sufficient to compute the weight of a heaviest path from the node i to 1 for $i = 1, 2, \ldots, n$.

Theorem 3.2 *Let $A = (a_{ij}) \in T_n(-\infty)$, $B = (b_{ij}) \in T_n(-\infty)$. Then $A \oplus B \in T_n(-\infty)$, $A \otimes B \in T_n(-\infty)$.*

Proof. Suppose $A = (a_{ij}) \in T_n(-\infty)$, $B = (b_{ij}) \in T_n(-\infty)$. It is clear that the matrix $A \oplus B \in T_n(-\infty)$.
Now, we assume that $A \otimes B = C = (c_{ij})$. Then $c_{ij} = \sum\limits_{k \in N}^{\oplus} a_{ik} \otimes b_{kj}$. We want to prove $c_{1j} = c_{1+r,j+r}$ for all $1 \le j$ and $j + r \le n$. So, we have
$c_{1j} = \sum\limits_{k \in N}^{\oplus} a_{1k} \otimes b_{kj} = \sum\limits_{1 \le k \le j}^{\oplus} a_{1-k} \otimes b_{k-j} = \sum\limits_{1 \le k \le j}^{\oplus} a_{1+r-(k+r)} \otimes b_{k+r-(j+r)} = \sum\limits_{1 \le k \le j}^{\oplus} a_{1+r,k+r} \otimes b_{k+r,j+r} = c_{1+r,j+r}$.
□

The structure of eigenspace of $-\infty$-triangular Toeplitz matrix requires the analysis of the eigenspace dimension. The next assertion describes this result which follows from the structure of D_A for $A = (a_{ij}) \in T_n(-\infty)$ and from *Theorem 1.2*.

Theorem 3.3 *Let $A = (a_{ij}) \in T_n(-\infty)$. Then $dim(Sp(A)) = n$.*

To compute all fundamental eigenvectors for a given matrix $A \in T_n(-\infty)$ the following procedure is applied ([15]):

Procedure : *Heaviest Path for Acyclic Digraph*

Input: D_A^{ac}, $A \in T_n(-\infty)$

Output: The vector $d(i)$ containing the weight of a heaviest path from

node 1 to node i for $i \in N$

1. Start $d(1) := 0$; $d(i) := -\infty$ for all other i; $a_{11} := -\infty$

2. For $i = 1$ to $n - 1$ do for each $j \in \Gamma(i)$ with $d(j) < d(i) + a_{ij}$ put

$$d(j) = d(i) + a_{ij}$$

3. End

The time complexity of the *Heaviest Path for Acyclic Digraph* procedure is $O(n^2)$ since each edge is considered only once.

Theorem 3.4 *Let $A \in T_n(-\infty)$ and $\lambda(A) = 0$. Then the matrix $\Delta(A)$ can be computed in $O(n^2)$ time.*

Proof. The first step of the algorithm tests whether the matrix $A = (a_{ij}) \in T_n(-\infty)$. It can be done in $O(n^2)$ steps. The second part of the algorithm uses the procedure for computing eigenvectors of A. By [6], it is sufficient to compute the eigenvalue $\lambda(A)$, then take the matrix A_λ, find the diagonal elements in $\Delta(A_\lambda)$ with the value $\xi_{ii} = 0$ and compute the i-th column of the metric matrix $\Delta(A_\lambda)$. Any such column x is an eigenvector to A_λ.

The eigenvalue $\lambda(A)$ and the matrix A_λ can be computed in $O(n^2)$ time, in view of *Corollary 3.1*. Since $A \in T_n(-\infty)$ is a given matrix due to *Theorem 3.2*, the metric matrix $\Delta(A_\lambda) \in T_n(-\infty)$ too, and from this it follows that for computing all elements of $\Delta(A_\lambda)$ it is sufficient to compute elements of the first row of $\Delta(A_\lambda)$.

The element ξ_{1j}, $j \in N$ of the metric matrix $\Delta(A_\lambda)$ is equal to the weight of the heaviest path from the node 1 to the node j in the corresponding digraph D_A^{ac}. It is clear that $\xi_{11} = 0$. By applying the *Heaviest Path for Acyclic Digraph* procedure to D_A^{ac} the output can be computed in $O(n^2)$ time.

In the remaining part of the proof, it suffices to emphasize that we use the *Heaviest Path for Acyclic Digraph* procedure only once, hence the complexity of computing all elements of $\Delta(A_\lambda)$ is $O(n^2)$.
□

From the definition of a $-\infty$-triangular Toeplitz matrix it follows that some elements of an eigenvector can be equal to $-\infty$ but each eigenvector is different from $(-\infty, \ldots, -\infty)^T$.

We summarize our previous results in the following:

Theorem 3.5 *There exists an algorithm $\mathscr{A}$, which, for a given matrix $A \in T_n(-\infty)$, computes the eigenvalue and the eigenvectors in $O(n^2)$ time.*

4. IRREDUCIBLE CASE: EIGENPROBLEM OF AN ε-TRIANGULAR TOEPLITZ MATRIX

This section presents the $O(n^2)$ algorithm for the case when a matrix A is an irreducible ε-triangular Toeplitz, i.e.

$A = (a_{ij}) \in T_n(\varepsilon)$ and $-\infty < \varepsilon \leq -(n-1)\max_{i\leq j}|a_{ij}|$. The case $\varepsilon = 0$ is analyzed in [18] and the suggested method is completely different from the algorithm.

Theorem 4.1 *Let $A = (a_{ij}) \in T_n(\varepsilon)$. Then $\lambda(A) = b_0 = 0$.*

Proof. Suppose that $A = (a_{ij}) \in T_n(\varepsilon)$ and $\sigma = (i_1,\dots,i_s)$ is a cycle. In case that $s > 1$ the cycle σ contains at least one lower diagonal element, say $a_{i_1i_2}$. Then $w(\sigma)/s = (a_{i_1i_2} + \dots + a_{i_si_1})/s \leq (\varepsilon + (s-1)\max_{i\leq j}|a_{ij}|)/s \leq 0$.

The assertion follows from the fact that the only cycle of which the cyclic means are equal to 0 are loops and the cycle $\pi = (1,\dots,n)$ in case that $b_1 = \max_{i\in N} b_i = \max_{i\in N}|b_i|$. Then $\lambda(A) = \max(0, \frac{b_1(n-1)+\varepsilon}{n}) = \max(0, \frac{a_{12}+\dots+a_{n-1,n}+\varepsilon}{n}) = 0$. The assertion now follows from the fact that the mean weight of the loops in D_A is 0.
□

Theorem 4.2 *Let $A = (a_{ij}) \in T_n(\varepsilon)$. Then*

(i) If $i < j$ then there exists a heaviest path $p = (i = i_1,\dots,i_k = j)$ in $D_{T_n(\varepsilon)}$ containing no edge (i_l,i_{l+1}) such as $i_l > i_{l+1}$,

(ii) If $i > j$ then each heaviest path $p = (i = i_1,\dots,i_k = j)$ in $D_{T_n(\varepsilon)}$ contains exactly one edge (i_l,i_{l+1}) such that $i_l > i_{l+1}$.

Proof. (i) Suppose that $A = (a_{ij}) \in T_n(\varepsilon)$ and a path $p = (i = i_1,\dots,i_k = j)$, $1 \leq k \leq n$, contains the edge (i_l,i_{l+1}) such that $i_l > i_{l+1}$. Then we obtain $w(p) = a_{i_1i_2} + \dots + a_{i_li_{l+1}} + \dots + a_{i_{k-1}i_k} = a_{i_1i_2} + \dots + \varepsilon + \dots + a_{i_{k-1}i_k} \leq -(n-1)\max_{i\leq j}|a_{ij}| + (k-2)\max_{i\leq j}|a_{ij}| = (k-n-1)\max_{i\leq j}|a_{ij}| \leq -\max_{i\leq j}|a_{ij}| \leq a_{i_1i_k}$ and instead of p we can take a path $p' = (i,j)$.

(ii) First, we assume that $p = (i = i_1,\dots,i_k = j)$ contains two adjacent edges, say, $(i_1,\ i_2)$ and $(i_2,\ i_3)$ such that $i_1 > i_2 > i_3$. Then $w(p) < w(p')$, where $p' = (i_1,i_3,\dots,i_k)$.

Now we suppose that no two edges (i_l,i_{l+1}), (i_s,i_{s+1}) such as $i_l > i_{l+1}$, $i_s > i_{s+1}$ of p are adjacent (w.l.o.g. $i_l > i_{s+1}$). Thus, we can decompose p by ε edges into three sequences of edges the weights of which are greater than ε. Then we have $w(p) = w(i_1,\dots,i_l,i_{l+1},\dots,i_s,i_{s+1},\dots i_k) = a_{i_1i_2} + \dots + a_{i_li_{l+1}} + \dots + a_{i_si_{s+1}} + \dots + a_{i_{k-1}i_k} \leq a_{i_1i_2} + \dots + a_{i_li_{s+1}} + \dots + a_{i_{k-1}i_k} = w(p'')$, where $p'' = (i_1,\dots,i_l,i_{s+1},\dots,i_k)$.
□

Since the eigennodes of $A \in T_n(\varepsilon)$ lie either on the loops or on the cycle $\sigma = (1,\dots,n)$ the next assertion describing the dimension of eigenspace directly follows.

***Corollary* 4.1** *Let $A = (a_{ij}) \in T_n(\varepsilon)$. Then $dim(Sp(A)) \in \{1,n\}$.*

To compute all the eigenvectors for a given matrix $A \in T_n(\varepsilon)$ we use the following results.

The matrix D=(d_{ij}), $d_{ij} = d_{j-i+1}$ is the $-\infty$-triangular matrix, where d_{j-i+1} is output of the *Heaviest Path for Acyclic Digraph* procedure, i.e.

$$\mathbf{D} = (d_{ij}) = \begin{pmatrix} 0 & d_2 & d_3 & \dots & d_{n-1} & d_n \\ -\infty & 0 & d_2 & \dots & d_{n-2} & d_{n-1} \\ -\infty & -\infty & 0 & \dots & & d_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\infty & -\infty & -\infty & \dots & 0 & d_2 \\ -\infty & -\infty & -\infty & \dots & -\infty & 0 \end{pmatrix}.$$

Now we will define the auxiliary vectors $u = (u_1,\dots,u_n)^T$ and $v = (v_1,\dots,v_n)^T$ as follows:

$$u_i = \max_{i\leq j} d_{ij} \text{ for } i = 1,\dots,n; \quad v_j = \max_{i\leq j} d_{ij} \text{ for } j = 1,\dots,n.$$

Theorem 4.3 *Let $A \in T_n(\varepsilon)$ and $\lambda(A) = 0$. Then the matrix $\Delta(A) = (\xi_{ij})$ can be computed in $O(n^2)$ time as follows:*

$$\xi_{ij} = \begin{cases} d_{ij} \text{ for all } i \leq j \\ u_i + \varepsilon + v_j \text{ for all } i > j. \end{cases}$$

Proof. The elements ξ_{ij} of the metric matrix $\Delta(A_\lambda)$ is equal to the weight of the heaviest path from the node i to the node j in the corresponding digraph D_A. From *Theorem 4.2 (i)* follows that elements of $\Delta(A_\lambda)$ lying over the diagonal are not equal to ε and just one diagonal and the lower diagonal elements is equal to ε due to *Theorem 4.2 (ii)*. As a consequence the over diagonal elements can be computed by *Heaviest Path for Acyclic Digraph* procedure for D_A^{ac}.
CLAIM: Let $i > j$. Then $\xi_{ij} = u_i + \varepsilon + v_j$.
Proof of CLAIM. Suppose that $i > j$. From *Theorem 4.2 (ii)* we know that each heaviest path $p = (i = i_1,\dots,i_k = j)$ in $D_{T_n(\varepsilon)}$ contains exactly one edge (i_l,i_{l+1}) such that $i_l > i_{l+1}$. The heaviest path from vertex i to the vertex j contains two parts $p_1 = (i,i_2,\dots,i_l)$ and $p_2 = (i_{l+1},\dots,i_{k-1},j)$ whereby no edge from p_1, p_2 has weight equal to ε. From the structure of p it is clear that $i \leq i_l$ and $i_{l+1} \leq j$ and the weight of p is maximal if and only if weights of p_1 and p_2 are maximal. Then we have $w(p_1) = u_i = d_{ii_l}$, $w(p_2) = v_j = d_{i_{l+1}j}$ and the assertion follows.
□

We summarize our previous results in the following:

Theorem 4.4 *There exists an algorithm $\mathscr{A}$ which for a given matrix $A = (a_{ij}) \in T_n(\varepsilon)$ computes the eigenvalue and the fundamental eigenvectors in $O(n^2)$ time.*

Proof. The checking of the ε-triangularity and the Toeplitz property is possible in $O(n^2)$ steps. The second part of the algorithm uses the procedure for computing eigenvectors of A.

The eigenvalue $\lambda(A)$ and the matrix A_λ can be computed in $O(n^2)$ time and due to the *Theorem 4.3* the metric matrix $\Delta(A_\lambda)$ can be computed in $O(n^2)$ steps.
□

Example. Let a matrix A have the form

$$\mathbf{A} = \begin{pmatrix} 0 & 2 & 1 & 7 & 2 & 7 \\ -35 & 0 & 2 & 1 & 7 & 2 \\ -35 & -35 & 0 & 2 & 1 & 7 \\ -35 & -35 & -35 & 0 & 2 & 1 \\ -35 & -35 & -35 & -35 & 0 & 2 \\ -35 & -35 & -35 & -35 & -35 & 0 \end{pmatrix}.$$

Using the *Heaviest Path for Acyclic Digraph* procedure we obtain the values $(d_1,d_2,d_3,d_4,d_5,d_6) = (0,2,4,7,9,11)$ and the matrix D looks like this

$$\mathbf{D} = \begin{pmatrix} 0 & 2 & 4 & 7 & 9 & 11 \\ -\infty & 0 & 2 & 4 & 7 & 9 \\ -\infty & -\infty & 0 & 2 & 4 & 7 \\ -\infty & -\infty & -\infty & 0 & 2 & 4 \\ -\infty & -\infty & -\infty & -\infty & 0 & 2 \\ -\infty & -\infty & -\infty & -\infty & -\infty & 0 \end{pmatrix}.$$

The auxiliary vectors $u = (u_1,\dots,u_n)^T$ and $v = (v_1,\dots,v_n)^T$ are equal to:

$$u = (11,9,7,4,2,0)^T,\ v = (0,2,4,7,9,11)^T.$$

Thus,

$$\xi_{ij} = \begin{cases} d_{ij} & \text{for all } i \le j \\ u_i + \varepsilon + v_j & \text{for all } i > j \end{cases}$$

and the metric matrix $\Delta(A_\lambda)$ takes the form:

$$\Delta(\mathbf{A}_\lambda) = \begin{pmatrix} 0 & 2 & 4 & 7 & 9 & 11 \\ -26 & 0 & 2 & 4 & 7 & 9 \\ -28 & -26 & 0 & 2 & 4 & 7 \\ -31 & -29 & -27 & 0 & 2 & 4 \\ -33 & -31 & -29 & -26 & 0 & 2 \\ -35 & -33 & -31 & -28 & -26 & 0 \end{pmatrix}.$$

In this case $dim(Sp(A)) = 6$.

REFERENCES

[1] P. Butkovič and R. A. Cuninghame-Green, An $O(n^2)$ algorithm for the maximum cycle mean of an $n \times n$ bivalent matrix, Discrete Applied Mathematics 35 (1992), pp. 157-162.

[2] P. Butkovič and R. A. Cuninghame-Green, Extremal Eigenproblem for Bivalent Matrices, Linear algebra and its applications (1994), pp. 1-13.

[3] P. Butkovič, S. Gaubert and R. A. Cuninghame-Green, Reducible spectral theory with applications to the robustness of matrices in max-algebra, The University of Birmingham, preprint 2007/16.

[4] J. Cochet-Terrason, G. Cohen, S. Gaubert, M. McGettrick and J.-P. Quadrat, Numerical computation of spectral elements in max-plus algebra, in: IFAC Conference on System Structue and Control (1998).

[5] R. A. Cuninghame-Green, Describing industrial processes with interference and approximating their steady-state behavior, Oper. Res. Quart. 13 (1962), pp. 95-100.

[6] R. A. Cuninghame-Green, Minimax algebra, Lecture Notes in Econ. and Math. Systems 166, Springer-Verlag, Berlin (1979).

[7] E. W. Dijkstra, A note on two problems in connection with graphs, Numer. Mathematik 1 (1959) pp. 269-271.

[8] M. Gavalec and J. Plavka, $O(n^2)$ algorithm for maximum cycle mean of Monge matrices in max-algebra, Discrete Appl. Math. 127 (2003), pp. 651-656.

[9] M. Gavalec and J. Plavka, Computing an eigenvector of a Monge matrix in max-plus algebra, Mathematical Methods of Operations Research 62 (2006), pp. 543 - 551.

[10] S. Gaubert, Théorie des systémes linaires dans les diodes, Thse, Ecole des Mines de Paris, 1992.

[11] M. Gondran and M. Minoux, Valeurs propres et vecteur propres dans les diodes et leur interprtation en thorie des graphes, Bulletin de la direction des etudes et recherches, Serie C, Mathematiques et Informatiques, No 2, (1977) pp. 25-41.

[12] B. Heidergott, G. J. Olsder and J. van der Woude, Max plus at work. Modeling and analysis of synchronized systems, Princeton University Press, 2004.

[13] R. M. Karp, A characterization of the minimum cycle mean in a digraph, Discrete Math. 23 (1978), pp. 309-311.

[14] E. L. Lawler, Combinatorial Optimization: Networks and Matroids, Holt, Rinehart and Wilston 1976.

[15] H. Noltemeier, Graphentheorie mit Algorithmen und Anwendungen Berlin, Walter de Gruyter, 1976.

[16] J. Plavka, Eigenproblem for circulant matrices in max-algebra, Optimization 50 (2001), pp. 477-483.

[17] J. Plavka, Eigenproblem for Monotone and Toeplitz Matrices in a max-Algebra, Optimization 53 (2003), pp. 95-101.

[18] P. Szabó, A short note on the weighted sub-partition mean of integers, Operations Research Letters (2009)

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