THE BIT ERROR PROBABILITY OF PSK SYSTEM IN THE PRESENCE OF INTERFERENCE AND NOISE

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SUMMARY

In this paper the bit error probability rather than the lower bound for two interferences is evaluated. The signal, interferences and noise are applied at the input to the phase-coherent communication receiver and an expression for the bit error probability is derived for the case when the reference carrier is not ideal. The analyzed model of the phase locked loop (PLL) is nonlinear. The reference carrier is extracted by the first and second order loop.

Keywords: *Phase-coherent communication receiver, Phase locked loop (PLL), Gaussian noise, Interference, Error probability*

1. INTRODUCTION

An expression for the bit error probability was calculated when the signal and additive Gaussian noise are applied to the input of the phase-coherent communication receiver with the phase locked loop [2]. Performance of the Coherent phase shift keying (CPSK) signals in the presence of noise and interferences have been considered in a number of papers [7,8]. In the reference [4] the lower bounds for digital communications with multiple interferences were determined. In this paper the bit error probability for two interferences is calculated. The performance of the phase-coherent communication receiver when the reference carrier is extracted by the phase locked loop (PLL) is determined.

2. SYSTEM MODEL

The error probability is derived when the detection of binary phase modulated signal is coherent. The model of the receiver for this case is given at Fig.1 [2]. The binary signal in the transmitter, η_k, which brings the information, is given in the form:

$$\eta_k(t) = \sqrt{2P}\sin\left[\omega_0 t + (\cos^{-1} m)x_k(t)\right],$$
$$(k = 1,2) \qquad (1)$$

where P is the total transmitted power, m is the coefficient which total power divides between the carrier and the lateral bands and $x_k(t), 0 \le t \le T$, is the binary signal which brings the information. The case $x_k(t) = \pm 1$ is of the greatest practical interest.
The received signal is in the form:

$$\Psi(t) = \sqrt{2P}\sin\left[\omega_0 t + (\cos^{-1} m)x_k(t) + \theta(t)\right] + n(t) \qquad (2)$$

where *θ(t)* is the random phase movement produced in the channel and *n(t)* is the Gaussian noise. This signal is demodulated in the receiver shown at Fig. 1.

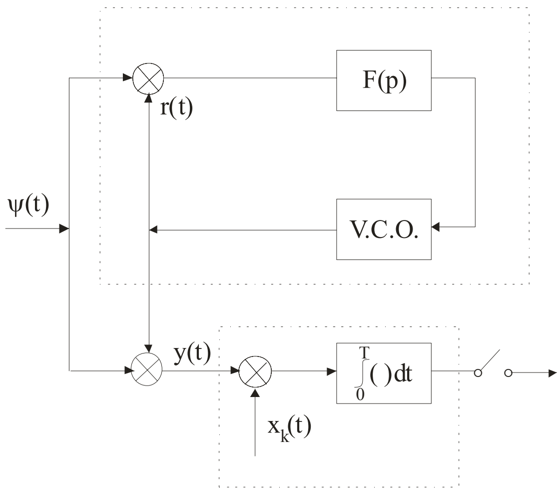

Fig. 1 The system model

The sinphase loop exists in the circuit for the carrier extraction. The loop filter is not ideal and it is of the first order. The referent carrier, obtained at the output of the extraction circuit, is:

$$r(t) = \sqrt{2}\cos\left[\omega_0 t + \Theta(t)\right] \qquad (3)$$

where $\Theta(t)$ is the evaluated value of the phase movement formed in the channel. The product of the signals *r(t)* and $\Psi(t)$, when the double frequencies terms are neglected, is:

$$y(t) = \sqrt{S}x_k(t)\cos\varphi(t) + n'(t) \qquad (4)$$

where $S = (1-m^2)P$, $\varphi(t)$ is phase error process and $n'(t)$ is the Gaussian noise with single-sided power density spectrum N_0 in W/Hz.
The decision is based on:

$$q=\int_0^T y(t)\left[x_1(t)-x_2(t)\right]dt \tag{5}$$

If $q \geq 0$ it can be taken that the transmitter sent the signal $x_1(t)$, if $q<0$ the sent signal is $x_2(t)$. In the time interval of one digit, T, the conditional probability density function of q, for given φ, is normal. The mean value and the variance of this distribution are:

$$M_k=(-1)^k 2\sqrt{S}\int_0^T \cos\varphi(t)dt$$
$$\sigma_k^2=2TN_0 \tag{6}$$

where $k=1,2$. From (6) we can obtain the conditional probability density function of q for both of hypothesis:

$$p(q_1/\varphi)=\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{(q_1-\sqrt{2}M_1/\sigma_1)^2}{2}\right]$$
$$p(q_2/\varphi)=\frac{1}{\sqrt{2\pi}}\exp\left[-\frac{q_2^2}{2}\right] \tag{7}$$

The k subscript on q corresponds to the hypothesis that $x_k(t)$ was transmitted $k=1,2$ [2].

The term for the conditional error probability for given φ is, from (7):

$$P_e/\varphi=Q(\sqrt{2RY})$$

where:

$$R=\frac{ST}{N_0}=\frac{E}{N_0}, \qquad Y=\frac{1}{T}\int_0^T \cos\varphi(t)dt,$$
$$Q(x)=\frac{1}{\sqrt{2\pi}}\int_x^\infty \exp\left(-\frac{z^2}{2}\right)dz$$

cos φ is taken to be constant in digit interval T [2].

3. ERROR PROBABILITY

Let the input to the phase-coherent communication receiver consist of the signal, interferences and additive Gaussian noise:

$$\begin{aligned} r(t)&=s(t)+i_1(t)+i_2(t)+n(t)=\\ &=A\cos\omega_0 t+A_1\cos(\omega_0 t+\theta_1)+\\ &+A_2\cos(\omega_0 t+\theta_2)+n(t)\end{aligned} \tag{8}$$

where: $s(t)$ is the signal, $i_1(t)$ is the interference, $i_2(t)$ is the co-channel interference, A is the signal amplitude, A_1 and A_2 are the interferences amplitudes, θ_1 and θ_2 are the interference phases and $n(t)$ is additive Gaussian noise.

The probability density functions of the phases θ_1 and θ_2 are:

$$p_1(\theta_1)=\begin{cases}\frac{1}{2\pi}, & |\theta_1|\leq\pi\\ 0, & |\theta_1|\rangle\pi\end{cases}$$
$$p_2(\theta_2)=\begin{cases}\frac{1}{2\pi}, & |\theta_2|\leq\pi\\ 0, & |\theta_2|\rangle\pi\end{cases} \tag{9}$$

Equation (8) may be written in the form:

$$r(t)=AR\cos(\omega_0 t+\psi)+n(t) \tag{10}$$

where

$$R=R(\cos\theta_1,\cos\theta_2)=$$
$$=\sqrt{1+\eta_1^2+\eta_2^2+2\eta_1\cos\theta_1+2\eta_2\cos\theta_2+2\eta_1\eta_2\cos(\theta_1-\theta_2)}$$
$$\psi=arctg\frac{\eta_1\sin\theta_1+\eta_2\sin\theta_2}{1+\eta_1\cos\theta_1+\eta_2\cos\theta_2}$$
$$\eta_1=\frac{A_1}{A}\quad \eta_2=\frac{A_2}{A} \tag{11}$$

ψ is the equivalent angle of signal and interference.

Under the assumption of a constant phase in the symbol interval, the conditional error probability for the phase-coherent communication system which uses the PLL to provide the synchronization is given by [2]:

$$P_{e/\theta_1,\theta_2,\phi}=Q\left(\sqrt{2R_b}\cos\phi\right) \tag{12}$$

where

$$Q(x)=\frac{1}{\sqrt{2\pi}}\int_x^{+\infty}\exp\left(-\frac{z^2}{2}\right)dz \tag{13}$$

$R_b=E/N_0$, E is the signal energy [3]; N_0 is the single-sided power density spectrum of the Gaussian noise in W/Hz and ϕ is the phase error process.

It has been shown [1] that the steady-state probability density function, of the modulo 2π reduced phase error is given with a good approximation by

$$p(\phi)=\frac{e^{\beta\phi+\alpha\cos\phi}}{4\pi^2 e^{-\pi\beta}\left|I_{j\beta}(\alpha)\right|^2}\int_\phi^{\phi+2\pi} e^{-\beta x-\alpha\cos x}dx \tag{14}$$

where $I_v(x)$ is the modified Bessel function of order v and argument x. The domain of definition for ϕ is any interval of width 2π centered about any lock point $2n\pi$, with n an arbitrary integer.

The parameters α and β that characterize equation (14), for the first order loop, are:

$$\alpha=\alpha_0 R$$
$$\beta=\beta_0\Omega \tag{15}$$

where α_0 and β_0 are constants [3, 11] and Ω is the frequency offset of the first term in equation (10). Hence

$$\Omega = \frac{d}{dt}\left(\omega_0 t + \psi\right) - \omega_0 = \frac{d\psi}{d\theta_1}\frac{d\theta_1}{dt} + \frac{d\psi}{d\theta_2}\frac{d\theta_2}{dt} \qquad (16)$$

Since $(d\theta_1/dt)=0$ and $(d\theta_2/dt)=0$, it follows $\Omega=0$, $\beta=0$ and equation (14) takes the form [9]:

$$p\left(\phi\right) = \frac{e^{\alpha_0 R\cos\phi}}{2\pi I_0\left(\alpha_0 R\right)} \qquad (17)$$

$p(\phi)$ is the probability density function (pdf) of the phase error in the form of a Tikhonov distribution.
Substituting $R_b=R_1R^2$ in equation (12), where R_1 corresponds to the case when there is no interferences, the conditional bit error probability, given for ϕ, θ_1 and θ_2 is determined. The average error probability is found averaging over all ϕ and over all θ_1 and θ_2:

$$P_e = \frac{1}{16\pi^3}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} Q\left(\sqrt{2R_1}R\cos\phi\right)\frac{e^{\alpha_0 R\cos\phi}}{I_0\left(\alpha_0 R\right)}d\theta_1 d\theta_2 d\phi \qquad (18)$$

If we substitute $x=\cos\phi$, $y=\cos\theta_1$ and $z=\cos\theta_2$ we obtain [see Appendix]

$$P_e = \frac{1}{16\pi^3}\int_{-1}^{1}\frac{dz}{\sqrt{1-z^2}}\int_{-1}^{1}\frac{dy}{\sqrt{1-y^2}}\int_{-1}^{1}\frac{e^{\alpha_0 Rx}}{\sqrt{1-x^2}}\frac{Q\left(\sqrt{2R_1}Rx\right)}{I_0\left(\alpha_0 R\right)}dx \qquad (19)$$

where

$$R = R\left(y,z\right) = \sqrt{1+\eta_1^2+\eta_2^2+2\eta_1 y+2\eta_2 z+2\eta_1\eta_2 yz+2\eta_1\eta_2\sqrt{1-y^2}\sqrt{1-z^2}} \qquad (20)$$

In order to calculate the bit error probability P_e we will apply the Gauss-Chebyshev quadrature formulas in N points.

Equation (19) can be reduced to [5]

$$P_e = P_e\left(N\right) = \frac{1}{N^3}\sum_{j=1}^{N}\sum_{m=1}^{N}\frac{1}{I_0\left(\alpha_0 r_{jm}\right)}\sum_{k=1}^{N} Q\left(\sqrt{2R_1}r_{jm}x_k\right)e^{\alpha_0 r_{jm}x_k} \qquad (21)$$

where x_k denotes the zeros of the Chebyshev polynomial $T_N(x)$.

$$x_k = \cos\frac{\pi}{2N}\left(2k-1\right) \quad k=1,\dots N \qquad (22)$$

and

$$r_{jm} = R\left(x_j, x_m\right) = \sqrt{1+\eta_1^2+\eta_2^2+2\eta_1 x_j+2\eta_2 x_m+2\eta_1\eta_2 x_j x_m+2\eta_1\eta_2\sqrt{1-x_j^2}\sqrt{1-x_m^2}}$$
$$j=1,\dots,N, \quad m=1,\dots,N \qquad (23)$$

The convergence of the Gauss quadrature formulas ($P_e(N)\rightarrow P_e$ when $N\rightarrow+\infty$) means that the method for calculating the bit error probability with the accuracy $\varepsilon=10^{-6}$ is based on the constructions of the sequence $\{P_e(N)\}$ (N=6,7,…) and application of the Δ^2-process in order to accelerate the convergence of this sequence [9]. The process is terminated when the difference between two successive terms of the accelerated sequence is less than ε.

For the case when the second order PLL is used, the error probability also can be derived. The procedure is similar to previous case, only the parameters α and β are defined as [2]

$$\alpha = \frac{r_1+1}{r_1}\rho - \frac{1-F}{r_1\delta_G^2}$$
$$\beta = \left(\frac{r_1+1}{r_1}\right)^2\frac{\rho}{F}\left[\frac{\Omega_0}{AK} - \left(1-F\right)\overline{\sin\varphi}\right]$$
$$\Omega_0 = \omega - \omega_0 \quad F = \tau_2/\tau_1 \quad r_1 = AK\,\tau_2^2/\tau_1$$
$$\delta_G^2 = \overline{\sin^2\varphi} - \overline{\left(\sin\varphi\right)^2} \qquad (24)$$

where AK is the loop gain, $\rho=2P_c/N_0W_L$ is the signal/noise ratio in the loop bandwidth, P_c is the carrier power in the auxiliary synchronizing channel, $W_L=(r_1+1)/(2\tau_2)$ ($r_1\tau_1>>\tau_2$) is the loop bandwidth, φ is the phase error process of the second order PLL; $\overline{\sin u}$ is the mean value of $\sin u$. Parameters τ_1 and τ_2 are defined by the loop transfer function

$$F\left(s\right) = \frac{1+\tau_2 s}{1+\tau_1 s} \qquad (25)$$

4. NUMERICAL RESULTS

Fig. 2 shows the bit error probability as a function of R_1 for various values of the parameter η_2 and with α_0=10dB and η_1=0.3.

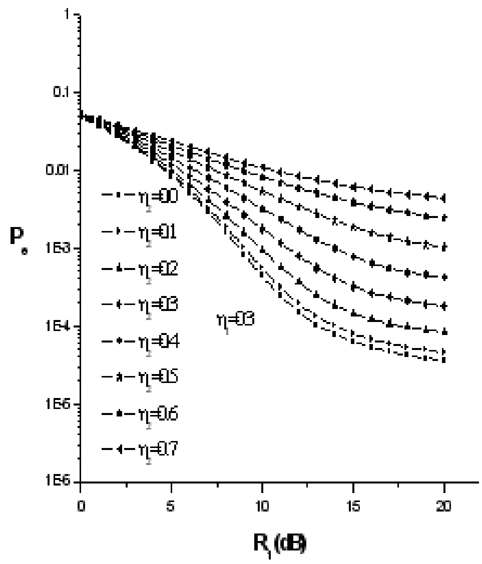

Fig. 2 The bit error probability against R_1 for various η_2

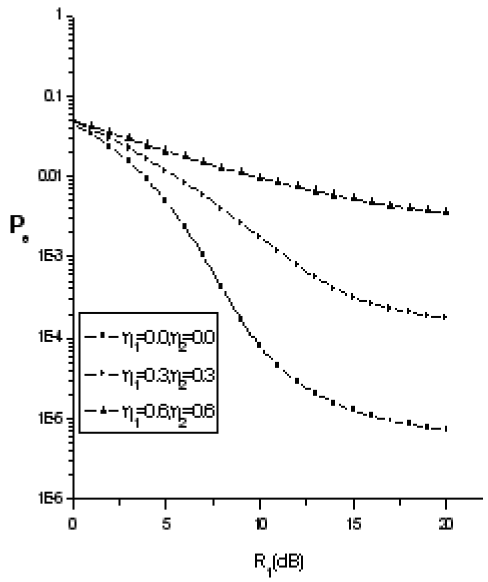

Fig. 3 The bit error probability against R_1 for various pairs of η_1, η_2

Fig. 3 shows the bit error probability as a function of R_1 for some pairs of values of the parameters η_1 and η_2. The curve for $\eta_1=\eta_2=0$ (no interferences) corresponds to well known results [6]. For R_1<10dB and $\eta_1, \eta_2 \geq 0.3$ the bit error probability is larger than 10^{-3}.

5. CONCLUSION

In this paper the bit error probability of PSK system in the presence of non-ideal extraction of the reference carrier, Gaussian noise and interferences is calculated. The circuit for the extraction of the reference carrier consists of the phase locked loop. The Gaussian noise and interferences have the influence to the reference carrier phase error. They appear in the circuit for the extraction of the reference carrier and at first input of the multiplier in the correlator, too. In this way they influence on the bit error probability expansion. The analyzed model of the phase locked loop (PLL) is nonlinear. We used the numerical methods for the calculation of the triple integral by the Gauss-Chebyshev quadrature formulas in N points. N is taken for the accuracy given. The obtain results can be applied in the PSK system design.

APPENDIX

In this section we give one application of Gaussian quadrature rules where is very important to calculated integrals with a high precision. We consider now the integral [5],

$$P_e = \frac{1}{\pi^m} \int_0^{\pi} \ldots \int_0^{\pi} Q\left[c\left(1 + \sum_{k=1}^{m} c_k \cos\theta_k \right) \right] d\theta_1 \ldots d\theta_m$$

where c and c_k are positive constants and the function $Q(t)$ is defined by:

$$w(t) = Q(t) = \frac{1}{\sqrt{2\pi}} \int_t^{\infty} e^{-x^2/2} dx \qquad \text{(A1)}$$

In our calculation, we used the following approximation $\left(0 \leq t < +\infty\right)$

$$Q(t) = \left(a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5\right) e^{-t^2/2} + \varepsilon \qquad \text{(A2)}$$

where $x = 1/\left(1+pt\right)$, $p = 0.23164189$, and $\left|\varepsilon\right| \leq 0.75 \times 10^{-7}$. The coefficients a_k are given by

$$a_1 = 0.127414796$$
$$a_2 = -0.142248368$$
$$a_3 = 0.7107068705$$
$$a_4 = -0.7265760135$$
$$a_5 = 0.5307027145$$

In order to calculate P_e (the error probability), we put $x_k = \cos\theta_k$, ($k = 1, \ldots, m$). Then, we get

$$P_e = \frac{1}{\pi^m} \int_{-1}^{1} \frac{dx_1}{\sqrt{1-x_1^2}} \ldots \int_{-1}^{1} \frac{1}{\sqrt{1-x_1^m}} Q\left[c\left(1 + \sum_{k=1}^{m} c_k x_k \right) \right] dx_m \cdot$$

Applying the Gauss-Chebyshev quadrature formula

$$\int_{-1}^{1} \frac{f(t)}{\sqrt{1-t^2}} dt = \frac{\pi}{n} \sum_{\nu=1}^{n} f\left(\tau_\nu\right) + R_n\left(f\right) \qquad \text{(A3)}$$

where $\tau_\nu \left(\nu = 1, \ldots, n\right)$ are zeros of the Chebyshev polynomial $T_n\left(t\right)$, i.e., $\tau_\nu = \cos\frac{(2\nu-1)\pi}{2n}$, $\nu = 1, \ldots, n$, successively m times, we obtain

$$P_e = \frac{1}{n^m} \sum_{\nu_1=1}^{n} \ldots \sum_{\nu_m=1}^{n} Q\left[c\left(1 + \sum c_k \tau_{\nu_k} \right) \right] + E_n^{(m)} \qquad \text{(A4)}$$

where $E_n^{(m)}$ is the corresponding error. Notice that for $f \in C^{2n}\left[-1,1\right]$ the remainder $R_n\left(f\right)$ in (A3) can be represented in the form

$$R_n\left(f\right) = \frac{\pi}{2^{2n-1}\left(2n\right)!} f^{(2n)}\left(\xi\right) \quad \left(-1 < \xi < 1\right).$$

In order to estimate $E_n^{(m)}$ we take $f\left(t\right) = Q\left(a+bt\right)$ $\left(z = a+bt, a, b > 0\right)$. Then, for the remainder term in the Gauss-Chebyshev formula (A3) we get

$$r_n = R_n\left(f\right) = \frac{\sqrt{\pi} b^{2n}}{2^{3n-1}\left(2n\right)!} e^{-v^2} H_{2n-1}\left(v\right),$$

where $v = \left(a + b\xi\right)/\sqrt{2} \left(-1 < \xi < 1\right)$. Since

$$\left|H_{2n-1}(v)\right| \le |v| e^{v^2/2} \frac{(2n)!}{n!},$$

we conclude that

$$\left|r_n\right| \le \frac{\sqrt{\pi} b^{2n}}{2^{3n-1} n!} |v| e^{-v^2} \le \pi K_n b^{2n},$$

not depending on a. By induction, it can be proving:

THEOREM

For the remainder $E_n^{(m)}$ in (A4) the following estimate

$$\left|E_n^{(m)}\right| \le \frac{c^{2n}}{2^{3n-1} n! \sqrt{\pi e}} \sum_{k=1}^{m} c_k^{2n} \qquad \text{(A5)}$$

holds.

Thus, basing on (A4) we have a formula for numerical calculation of the integral P_e in the form

$$P_e \approx P_e^{(n)} = \frac{1}{n^m} \sum_{v_1=1}^{n} \cdots \sum_{v_m=1}^{n} Q\left[c\left(1 + \sum_{k=1}^{m} c_k \tau_{v_k}\right)\right] \qquad \text{(A6)}$$

If the error in (A2) is such that $|\xi| \le E$, then for the total error in the approximation (A6) we have:

$$\left|\xi_T\right| \le E + \left|E_n^{(m)}\right|.$$

The number of nodes in the Gauss-Chebyshev formula (A3) should be taken so that the upper bounds of the error $E_n^{(m)}$, given in (A5), are the same order as E.

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