ANALYSIS OF SOME NEW APPROXIMATIONS OF PIECEWISE UNIFORM POLAR QUANTIZATION

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SUMMARY

In this paper the piecewise uniform polar quantization of Gaussian source is analyzed. Simultaneous inside the rings after the first partition the constant probability density function of input signal vector amplitude is supposed. For this case and for the given code rate we optimized the granular distortion in order to get the manner of total points number distribution per rings after the first partition; than the manner of the second partition, i.e. we evaluated the expressions for amplitude levels number and the phase levels number on one amplitude level. Also we found the expression for granular distortion which we used to estimate the suggested model. Namely, we compare the obtained signal to quantization noise ratio with the known optimal ratio and on these bases we conclude, among the other things, under which condition the suggested approximation can be applied.

Keywords: *piecewise uniform polar quantization, high resolution analysis, signal to quantization noise ratio*

1. INTRODUCTION

During the two-dimension vector quantization, vector obtained by sampling input signal in two points is replaced with vector from allowed set of vectors in such way that the quantization error is the smalest. In the case of polar quantization, these vectors are represented with their amplitude and phase which we consider in this paper. Successful vector quantization depends on an appropriate choise of allowed vector set (codebook). Quantization is necessary step in the digitalization process, but there are difficulties which cause the quantization error unavoidable during this process. So the quantization should be performed on such way that the quantization error doesn't reflect on signal reconstruction. This means that the allowed vector set should be chosen that the mean square error (distortion) is minimal. Since the quantization error is function of random signal which carries information, the error is also random variable. Therefore, the excellent knowledge of signal statistical caracteristics such as the joint probability density function of input vector amplitude and phase is necessary.

Polar quantization techniques as well as their applications in areas such as computer holography, discrete Furrier transform encoding, image processing and communications have been studied extensively in the literature. Synthetic Aperture Radars (SARs) images can be represented in the polar format (i.e., magnitude and phase components) [1]. Uniform polar quantizers are employed in Synthetic Aperture Radars (SARs) imaging systems, interferometric and polarimetric applications [1,2]. Optimal uniform quantization is given in [3], but optimal quantizer is nonuniform. A generalization of uniform polar quantizer is a piecewise uniform polar quantizer. One of the most important results in polar quantization is due Swaszek and Ku who derived the asymptotically optimal nonuniform polar quantization [4]. The support region for scalar quantizers has been found in [5] by minimization of the total distortion D, which is a combination of granular (D_g) and overload (D_o) distortions, $D = D_g + D_o$. We perform the two-step optimization as in [6].

From the later presented facts, we can see the importance of codebook and the necessity of some algorithm for its determination. The procedure is as follows: the given bit rate determines the set dimension; after that the support region, the amplitude and the phase levels numbers are found by means of distortion optimization with simultaneous care about the signal statistical characteristics. That means that codebook depends not only on the chossen quantizer model, but also on the input signal.

The high resolution (asymptotic) analysis of distortion determines the theoretical boundary and gives the opportunity for the quantizer parameters deffinition. Also, the usefullnes of the proposed quantizer is estimated on the base of the comparission between the obtained distortion and known optimal distortion.

In this paper we will find the signal to quantization noise ratio of two-dimension piecewise uniform polar quantizer for Gaussian source and for the given bit rate. Simultaneous some approximations for the probability density function of vector amplitude are suqqested. The approximation is applied in order to simplify the quantizer construction. The comparison between the signal to quantization noise ratio obtained on this way and the known optimal signal to quantization noise ratio will show under which certain conditions the approximation application is correct.

The Gaussian source has the importance because of using Gaussian quantizer on an arbitrary source; we can take advantage of the central limit theorem and the known structure of an optimal scalar quantizer for a Gaussian random variable to encode a general process by first filtering it in order to produce an approximately Gaussian density, scalar-quantizing the result, and then inverse-filtering to recover the original [7].

2. HIGH RESOLUTION ANALYSIS OF SUGGESTED PIECEWISE UNIFORM POLAR QUANTIZATION

The piecewise uniform polar quantization presents the improvement of uniform polar quantization, but on the other hand it leads to more complex structure of the quantizer. The piecewise uniform polar quantization more successfully follows the statistical characteristics of signal. It is achieved with the granular region partition into L rings (first partition) inside which there are new equidistant rings (L_i) with amplitude step Δ_i (Fig. 1a). Indeed, we can consider that the piecewise uniform polar quantizer consists of L uniform polar quantizaters with their own amplitude step Δ_i. After the first partition the amplitude decision levels would be noted by r_i, $0 \le i \le L$, $r_0 = 0$, $r_L = r_{max}$, where r_{max} is granular region boundary. In general case, these decision levels don't have to be equidistant, but if there are than $r_{i+1} - r_i = \Delta = r_{max}/L$. Now, we can say that the amplitude step for partition i is $\Delta_i = (r_{i+1} - r_i)/L_i$, as well as that the amplitude decision and reproduction levels of the piecewise uniform polar quantizers are

$$r_{i,j} = r_i + (j-1)\frac{r_{i+1} - r_i}{L_i}, \quad 0 \le i \le L-1, \quad 1 \le j \le L_i + 1$$

$$m_{i,j} = r_i + \left(j - \frac{1}{2}\right)\frac{r_{i+1} - r_i}{L_i}, \quad 0 \le i \le L-1, \quad 1 \le j \le L_i \qquad (1)$$

respectively.

Also, in a general case the phase levels number at amplitude level j inside partition i ($N_{i,j}$) don't have to be the same on each level inside one partition (Fig. 1b). Such polar quantization is called also unbounded polar quantization. Here the points number inside one partition is $N_i = \sum_{j=1}^{L_i} N_{i,j}$, while the points number for the hole quantizer is $N = \sum_{i=1}^{L} N_i$. There is also another situation when the phase levels number is the same on all amplitude levels inside one partition (M_i). Than we have the product polar quantization because of the points number in partition is product of the amplitude levels number and the phase levels number ($N_i = L_i \times M_i$). We will observe the general case, i.e. we will consider the unbounded polar quantization where the decision and reproduction phase levels are

$$\phi_{i,j,k} = (k-1)\frac{2\pi}{N_{i,j}}, \quad 1 \le k \le N_{i.j}, 1 \le i \le L, 1 \le j \le L_i$$

$$\psi_{i,j,k} = (2k-1)\frac{\pi}{N_{i,j}}, \quad 1 \le k \le N_{i.j}, 1 \le i \le L, 1 \le j \le L_i \qquad (2)$$

respectively.

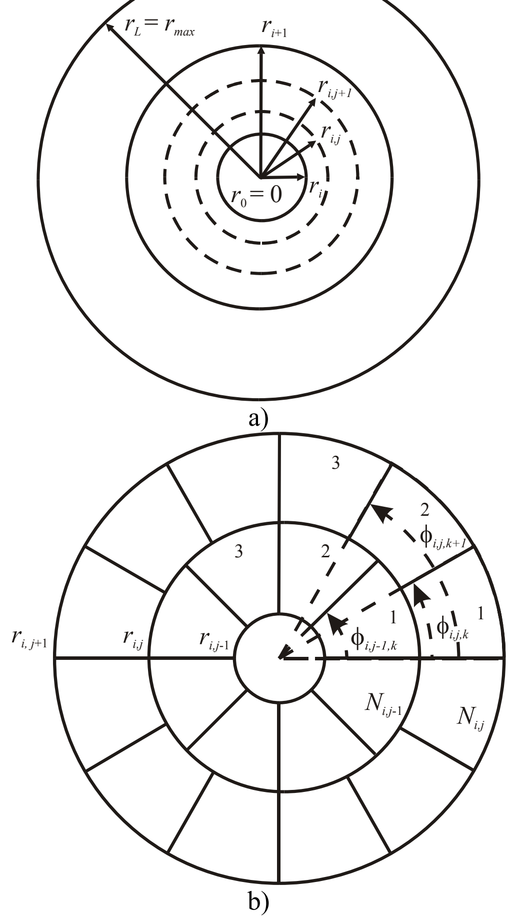


Fig. 1 Amplitude (a) and phase (b) levels at the piecewise uniform polar quantization

Since the piecewise uniform polar quantizer consists of L uniform polar quantizers which have the own distorzion $D(i)$, the total average mean square error (distortion) per one sample, i.e. one dimension is

$$D = D_g + D_o = \sum_{i=1}^{L} D(i) + D_o$$

$$D(i) = \frac{1}{2}\sum_{j=1}^{L_i}\sum_{k=1}^{N_{i,j}}\int_{\phi_{i,j,k}}^{\phi_{i,j,k+1}}\int_{r_{i,j}}^{r_{i,j+1}}\left[r^2 + m_{i,j}^2 - 2rm_{i,j}\cos\left(\phi - \psi_{i,j,k}\right)\right]f(r,\phi)\,dr\,d\phi \qquad (3)$$

$$D_o = \frac{1}{2}\sum_{k=1}^{N_{L,L_L}}\int_{\phi_{L,L_L,k}}^{\phi_{L,L_L,k+1}}\int_{r_{\max}}^{+\infty}\left[r^2 + m_{L,L_L}^2 - 2rm_{L,L_L}\cos\left(\phi - \psi_{L,L_L,k}\right)\right]f(r,\phi)\,dr\,d\phi$$

where D_g is total granular distortion, D_o overload distortion and $f(r,\phi)$ the joint probability density function of input vector amplitude and phase.

As we citied in the introduction, we will consider the case of piesewise uniform polar quantization of Gaussian source when the following approximation will be applied: the amplitude probability density function is replaced with constant inside the rings after the first partition. Without losing generality, we suppose that the input signal samples are independent Gaussian variables with zero mean value and variance 1, which means that the joint probability density function of vector amplitude and phase is $f(r,\phi) = r/(2\pi)\exp(-r^2/2)$, i.e. amplitude and phase of input vector are independent random variable. The input vector phase has the uniform distribution while the input vector amplitude has the Rayleigh probability density function. First we can solve integrals per ϕ and after that the distortion per sample for partition i is

$$D(i) = \frac{1}{2}\sum_{j=1}^{L_i}\int_{r_{i,j}}^{r_{i,j+1}}\left[\left(r - m_{i,j}\right)^2 - \frac{r m_{i,j}\pi^2}{3N_{i,j}^2}\right] f(r)dr\,, \quad (4)$$

where $f(r)$ is amplitude probability density function. Now, we apply approximation for amplitude probability density function. We suppose that $f(r)$ is constant inside partition i ($f(r) = f_i$), and later we will perform analysis for the next cases:

1st case: $f_i = r_i\exp(-r_i^2/2)$ for $r_i \le r \le r_{i+1}$

(the amplitude probability density function inside partition i has the value for lower ring boundary [8])

2nd case: $f_i = r_{i+1}\exp(-r_{i+1}^2/2)$ for $r_i \le r \le r_{i+1}$

(the amplitude probability density function inside partition i has the value for upper ring boundary)

3rd case:

$$f_i = \int_{r_i}^{r_{i+1}} r\exp(-r^2/2)dr = \exp(-r_i^2/2) - \exp(-r_{i+1}^2/2)$$

for $r_i \le r \le r_{i+1}$

(the amplitude probability density function inside partition i has average value)

As result of this assumption application we can write that the distortion per sample for partition i is

$$D(i) = \frac{f_i}{2}\sum_{j=1}^{L_i}\int_{r_{i,j}}^{r_{i,j+1}}\left[\left(r - m_{i,j}\right)^2 - \frac{r m_{i,j}\pi^2}{3N_{i,j}^2}\right]dr = \frac{f_i}{2}\left(\frac{\Delta_i^2}{12}P_i + \sum_{j=1}^{L_i}\frac{{m_{i,j}}^2\pi^2}{3N_{i,j}^2}\Delta_i\right), \quad (5)$$

where

$$P_i = \int_{r_i}^{r_{i+1}} dr = (r_{i+1} - r_i)\,. \quad (6)$$

The final expression for total granular distortion per sample is derived after optimization which consists of 3 steps. First we will optimize $D(i)$ per $N_{i,j}$ while we respect the limitation which is given by $\sum_{j=1}^{L_i} N_{i,j} = N_i$. This optimization is done by means of Lagrange multiplicator method. On this way we found that $D(i)$ is minimal for

$$N_{i,j}^{opt} = N_i\frac{\sqrt[3]{m_{i,j}^2}\,\Delta_i}{l_i}\,, \quad (7)$$

where

$$l_i = \int_{r_i}^{r_{i+1}}\sqrt[3]{r^2}\,dr = \frac{3}{5}\left(r_{i+1}^{5/3} - r_i^{5/3}\right). \quad (8)$$

The next step will be the optimization of the obtained expression $D(i)$ per L_i which gives the optimum for rings number inside partition i

$$L_i^{opt} = (r_{i+1} - r_i)\sqrt[4]{\frac{N_i^2(r_{i+1} - r_i)}{4\pi^2 l_i^3}}\,. \quad (9)$$

Now the expression for total granular distortion $D_g = \sum_{i=1}^{L} D(i)$ is minimized by N_i with limitation $\sum_{i=1}^{L} N_i = N$. We again apply Lagrange multiplicator method and for points number inside partition i we derived the next formula

$$N_i^{opt} = N\frac{\sqrt[4]{l_i^3 f_i^2 (r_{i+1} - r_i)}}{\sum_{j=1}^{L}\sqrt[4]{l_j^3 f_j^2 (r_{j+1} - r_j)}}\,. \quad (10)$$

After this the total granular distortion per sample is

$$D_g = \frac{\pi}{6N}\left(\sum_{i=1}^{L}\sqrt[4]{l_i^3 f_i^2 (r_{i+1} - r_i)}\right)^2. \quad (11)$$

If $r_{i+1} - r_i = const. = \Delta = r_{max}/L$, than the granular distortion will be

$$D_g = \frac{\pi\sqrt{\Delta}}{6N}\left(\sum_{i=1}^{L}\sqrt[4]{l_i^3 f_i^2}\right)^2. \quad (12)$$

Here we will present the equation for known optimal distortion of piecewise uniform polar quantizer [9] in order to compare the obtained results with reference ones

$$D_g^* = \frac{\pi}{6N}\left(\sum_{i=1}^{L}\sqrt[4]{(l_i^*)^3 P_i^*}\right)^2, \quad (13)$$

where

$$P_i^* = \int_{r_i}^{r_{i+1}} f(r)dr$$

$$l_i^* = \int_{r_i}^{r_{i+1}} \sqrt[3]{r^2 f(r)}dr$$

We will calculate the signal power to granular distortion ratio in dB and it will be called the signal to quantization noise ratio. Since the parameters of Gaussian process are normalized, the signal to quantization noise ratio will be

$$SNR_g = 10\log\frac{1}{D_g}, \quad (14)$$

where index g refers on granular distortion.

3. RESULT DISCUSSION

The results of the signal to quantization noise ratio are obtained by high resolution analysis with condition that the amplitude probability density function inside the ring after the first partition is constant. Here we have to define 2 parameters, code rate R and granular regione boundary r_{max}. The code rate or the vector quantizer rate is

$$R = \frac{1}{2}\log_2 N, \quad (15)$$

where N is the codebook size. The granular region boundary is calculated by equation

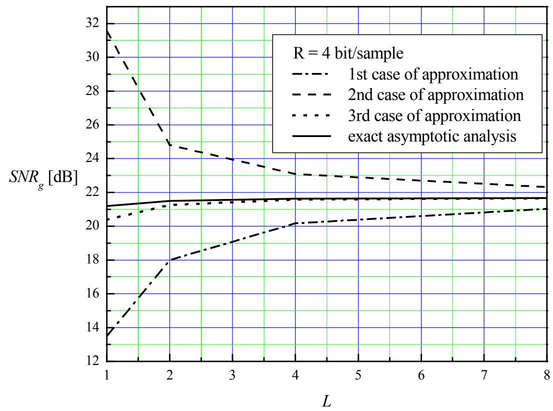
$$r_{\max} = 2\sqrt{\ln\sqrt{\frac{N}{2}}}, \quad N = 2^{2R}. \quad (16)$$

It is well known that total distortion with the granular region boundary determined by eq. (16) has minimal value [10]. The approximation of the amplitude probability density function is performed on 3 ways, for the lower ring boundary (1st case of approximation), for the upper ring boundary (2nd case of approximation) and with the average amplitude probability density function inside the ring (3rd case of approximation).

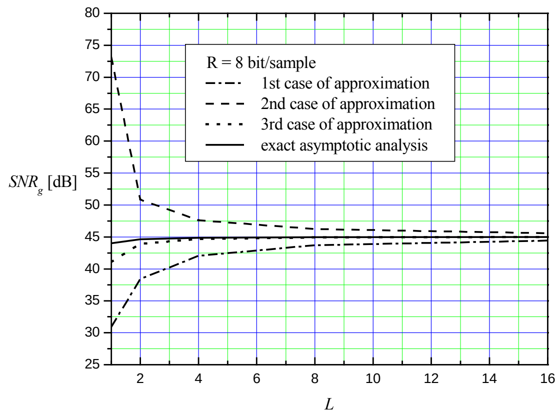
Fig. 2 and Fig. 3 present grafics of the obtained resuts. Also, the signal to quantization noise ratio calculated for exact asymptotic analysis (whithout approximation for amplitude probability density function) is also shown on these figures (solid lines). It is important to mention that even the second case of approximation gives the greatest signal to quantization noise ratio, it is not the best approximation. Namely, this approximation is very rude.

Fig. 2 shows the signal to quantization noise ratio dependence as a function of the value L for a given code rate R. Although the lines of all approximation cases approuch to solid line, the third case of approximation the best corespond to solid line, i.e. it has the smalest error. Namely, convergence of the second and the third approximations are slow, i.e. convergence is achived with L value which are unpossible for given R. This means that approximation can be applied only when the amplitude probability density function inside the ring is replaced with the average amplitude probability density function for the same ring.

It is obvious from the figure that uniform polar quantization (L=1) is not sutiable for any approximation. Evan for the third case of approximation and R = 8 bit/sample, the signal to quantization noise ratio value is less than the value of the appropiate point on solid line (44.0249 dB) for 2.9017 dB. So the approximation of amplitude probability density function can be apply if the amplitude step is variable along the granular region.



a)

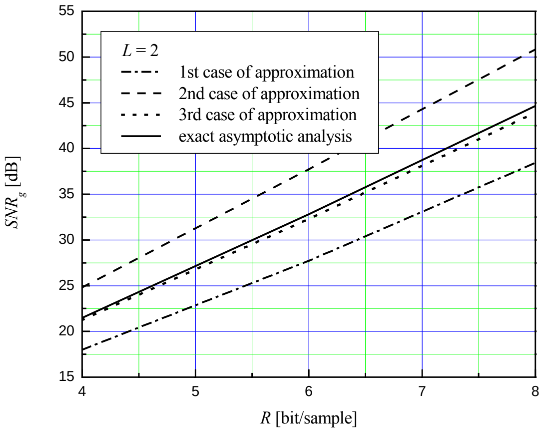


b)

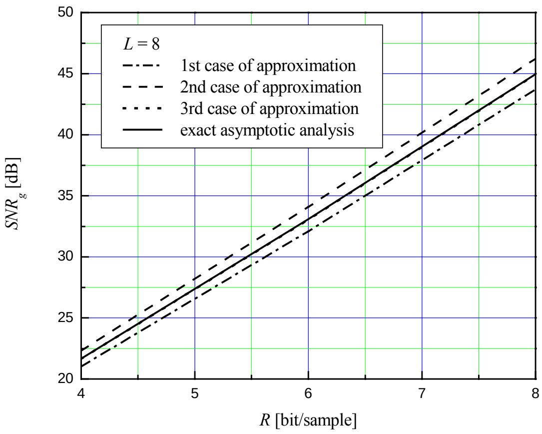
Fig. 2 Signal to quantization noise ratio as a function of number of rings after the first partition L for given code rate:
a) R = 4 bit/sample, b) R = 8 bit/sample

It is also interesting to observe from the Fig. 2a and 2b that the better results are achived for lesser R, i.e. for lesser size of codebook. This is more obvious in Fig. 3 on which the signal to quantization noise ratio as a function of R for a given L is presented. Namely, the lines that correspond to the cases with approximation are not parallel with line that correspond to the exact asymptotic analysis (solid line), i.e. the lines are closer to solid line for smaler

R although these deviations are not significant. Also, if we compare Fig. 3a and 3b, we again see that the better results of approximation aplication are achived for greater *L*, i.e. when the amplitude step severel times varies along granular region.



a)



b)

Fig. 3 Signal to quantization noise ratio dependence on code rate *R* for given number of rings after the first partition:
a) $L = 2$, b) $L = 8$

4. CONCLUSION

In this paper we analyze the piecewise uniform polar quantization for Gaussian source which is the process fundamental for the digitalization. In our research we used asymptotic method during which we applied new approximation of the amplitude probability density function. We suggested the analysis model at which the amplitude probability density function inside the rings after the first partition is replaced with constant on tree ways.

The results of the signal to quantization noise ratio obtained by asymptotic analysis showed that the third case of approximation, with average amplitude probability density function gives the best results. For given code rate the signal to quantization noise ratio tends to theoretical maximum with the increase of the number of rings after the first partition, i.e. it is better that the amplitude step severel times varies along granular region. It is also shown that for given number of rings after the first partition the better results of approximation application are achieved for lower code rate.

The other two approximations didn't give the expected results. Their application is theoretically possible for the number of rings after the first partition which is impossible in practice.

These results and the fact that we apply constant amplitude probability density function lead us to conclusion that the application of the third case of approximation is possible in practice which is significant for the easier construction of the piecewise uniform polar quantizer for Gaussian source.

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