BINARY ANALYSIS OF CENTER-WEIGHTED MEDIAN FILTERS

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SUMMARY

This paper focuses on the binary analysis of the center-weighted median (CWM) filters. This subclass of nonlinear stack filters is based on minimal positive Boolean functions (PBFs), the complexity of which depends on two parameters only, such as a window size and a weight of the central sample. It can be easily seen that the complexity of PBFs corresponding with CWMs increases with the increasing window size and the central weight. The above mentioned fact makes the implementation of CWMs in practical applications rather complicated.

In order to simplify the implementation, a new analysis, based on the determination of binary outputs according to the configuration of ones and zeros inside binary input sets, is provided. By this way, the outputs of binary filters are practically known at once with no PBF generation and the calculation of binary operations. The achieved simplification is described in the form of the elementary expressions, where the binary value of the central sample and a number of the ones in its neighbourhood play the key role. Thus, the proposed method makes the implementation of CWM filters more flexible, since it is possible to extend it for the arbitrary combination of the window size and the central weight.

Keywords: *stack filter, center-weighted median filter, positive Boolean function, binary analysis.*

1. INDRODUCTION

A number of nonlinear filters have been widely used in many signal and image smoothing applications since these filters effectively attenuate impulse or signal-depended noise. Additional support for the frequent use of many nonlinear filters lies in their definition in the binary domain such as it is in the case of a large family of stack filters [1],[5] that can be effectively implemented through the threshold decomposition architecture. Thus, the input signal is decomposed into a set of binary signals, the same filtering operation is performed at all binary levels and finally, an output sample is given by the summing up of binary filter outputs. If the stacking property is satisfied, the class of stack filters is based on positive Boolean functions (PBFs) [3],[4] that correspond with a filter operation in the real domain. Note is that the most efficient implementation of stack filters can be achieved through bit-serial implementation of PBFs [4].

Besides the well-known median filter, stack filters include [4] a number of more interesting filter classes such as rank-order filters, center-weighted medians (CWMs), weighted median (WM) filters and weighted order-statistic filters. Unlike the median filtering, a design of CWMs [6] provides larger flexibility with the possibility to achieve the balance between the signal-detail preservation and the noise suppression. The consideration of the central sample significance which is expressed by its weight, supports above capability. Of course, WM filters prove higher degree of the freedom than the CWMs, however, the CWMs are controlled by one parameter (the weight of the central sample), only.

For that reason, the implementation of CWMs is relatively simply. In addition, a new simplification based on the analysis of PBFs corresponding with CWMs is presented. Thus, this paper contributes to better understanding of the binary CWM filtering.

2. CWM OPERATION IN REAL DOMAIN

Let $x_1, x_2, ..., x_N$ characterise the input set determined by a filter window of a size N. Let $w_1, w_2, ..., w_N$ be a set of positive integer weights associated with the input samples so that w_i is associated with x_i for $i = 1, 2, ..., N$. The output of WM [4],[6] is defined by

$$y = med\{w_1 \lozenge x_1, w_2 \lozenge x_2, ..., w_N \lozenge x_N\} \tag{1}$$

where y is filter output, med is a median operator (it performs the selection of the central sample of an ordered set) and $\lozenge$ characterises a duplication operator given by

$$w_i \lozenge x_i = \underbrace{x_i, x_i, ..., x_i}_{w_i \text{ times}} \quad \text{for } i = 1, 2, ..., N \tag{2}$$

The filtering algorithm defined by (1) requires the sorting of input samples, the duplication (2) of each sample x_i (for $i = 1, 2, ..., N$) to w_i times and finally, the outputting the median value of the extended set. Besides the rank-order information (it is expressed by an ordered set), the temporal-order information of the input set $x_1, x_2, ..., x_N$ is considered (it is expressed by the weight vector), too.

Let only the central sample $x_{(N+1)/2}$ of the input set $x_1, x_2, ..., x_N$ be duplicated to $w_{(N+1)/2}$ times, whereas the other weights are retained equal to one, i.e.

$$w_i = \begin{cases} \{1,3,...,N\} & \text{for } i=(N+1)/2 \\ 1 & \text{otherwise} \end{cases} \quad (3)$$

then the output of CWM [6] is given by the simple rewriting (1) to

$$y = med\{x_1, x_2, ..., w_{(N+1)/2} \Diamond x_{(N+1)/2}, ..., x_{N-1}, x_N\} \quad (4)$$

From definition (3), it is clear that the central weight $w_{(N+1)/2}$ is forced to be an odd positive integer bounded by one and the window size N.

In general, the behaviour of CWM filter depends on the value of the central weight. An identity operation, i.e. no smoothing, is performed, when the central weight is equal to maximum value so that $w_{(N+1)/2} = N$. The additional increasing of the central weight $w_{(N+1)/2}$ is irrelevant, since the CWM filter will still perform an identity operation. If the central weight $w_{(N+1)/2}$ is between the minimum and the maximum value, then CWM filter provides the best balance between the noise suppression and the signal-details preservation. For the minimum value of the central weight, i.e. $w_{(N+1)/2} = 1$, the CWM filter is equivalent to a median filter.

3. CWM OPERATION IN BINARY DOMAIN

To perform the CWM filtering operation in binary domain, it is necessary to convert the CWM into its PBF form, where the conversion can be executed by two approaches.

The first one outcomes from the general WM conversion algorithm [6] including three steps such as

- List all the summations of weights that are greater than or equal to the threshold value

$$t = \frac{1}{2}(1 + \sum_{i=1}^{N} w_i) \quad (5)$$

- Represent each summation by a logical product with the following condition: If the weight w_i is presented in the summation, then the logical product contains sample x_i; otherwise, the logical product does not contain x_i.
- Minimise the disjunction of those logical products to a minimum sum of products form.

The second approach is valid especially for the case of CWM filters. The minimal PBF corresponding with the CWM of window length N and the central weight $w_{(N+1)/2}$ can be obtained by following algorithm [2]:

- Create logical products of $(N - w_{(N+1)/2} + 2)/2$ samples that contain the central sample $x_{(N+1)/2}$.
- Create logical products of $(N + w_{(N+1)/2})/2$ samples without the central sample $x_{(N+1)/2}$.
- Finally, logical sum up the logical products of previous steps.

By any means, the second conversion between CWMs and PBFs is easier to understand than the Boolean expression of CWMs through the first algorithm, since the second algorithm utilises natural dependence on the central weight $w_{(N+1)/2}$, only. In addition, it is very advantageous to separate the minimal PBF of CWM filters according to the second conversion algorithm.

4. PROPOSED ANALYSIS

In order to provide the additional analysis, it is necessary to present the advantage of PBFs expressions that lies in their implementation through the MIN-MAX realisation. The MIN operator represents the conjunction of binary samples and on the other hand, the MAX operator expresses the disjunction of binary samples. For example, the PBF of three-samples median [6] is given by

$$f_{med}(x_1, x_2, x_3) = x_1 x_2 + x_1 x_3 + x_2 x_3 \quad (6)$$

The same operation can be expressed by MIN and MAX operators, i.e. it results in

$$med\{x_1, x_2, x_3\} = \mathrm{MAX}\{\mathrm{MIN}\{x_1, x_2\}, \mathrm{MIN}\{x_1, x_3\}, \mathrm{MIN}\{x_2, x_3\}\} \quad (7)$$

Now, the above-mentioned approach is incorporated to PBFs corresponding with CWMs. It is clear that the complexity of PBFs increases with increased values of the central weight and the window length. For that reason, it is very ineffective to generate the PBFs and compute consistently their output. The proposed simplification lies in the output determination in the dependence on the configuration of the binary input samples.

Let $f_{B1}(.)$ be a Boolean function expressed through logical summing of logical products having $(N - w_{(N+1)/2} + 2)/2$ samples including the central sample $x_{(N+1)/2}$ and $f_{B2}(.)$ a Boolean function given by logical summing of logical products of $(N + w_{(N+1)/2})/2$ neighbouring samples. The minimal PBF $f_{PBF}(.)$ corresponding with the CWM of the window length (size) N and the central weight $w_{(N+1)/2}$ is given by

$$f_{PBF}(.) = f_{B1}(.) + f_{B2}(.) \quad (8)$$

The above equation can be equivalently expressed as

$$f_{PBF}(.) = \mathrm{MAX}\{f_{B1}(.), f_{B2}(.)\} \quad (9)$$

and the output of $f_{PBF}(.)$ results in 1 if and only if $f_{B1}(.)$ or $f_{B2}(.)$ is equal to 1, otherwise $f_{PBF}(.) = 0$. Now, let us analyse Boolean functions $f_{B1}(.)$ and $f_{B2}(.)$, separately.

According to logical products of $(N - w_{(N+1)/2} + 2)/2$ samples that contain the central sample $x_{(N+1)/2}$, the output value of $f_{B1}(.)$ depends on the central sample $x_{(N+1)/2}$. If $x_{(N+1)/2} = 0$, $f_{B1}(.)$ results in 0, because each logical product of $f_{B1}(.)$ expressed as

$$\underset{i_j \in \Im}{\mathrm{MIN}}\{x_{(N+1)/2}, x_{i_1}, x_{i_2}, ..., x_{i_{(N-w_{(N+1)/2})/2}}\} \quad (10)$$

contains the central sample $x_{(N+1)/2}$. Note is that the constraint $\Im$ is defined by

$$1 \le i_1 \le i_2 \le ... \le i_j \le ... \le i_{(N-w_{(N+1)/2})/2} \le N$$

$$i_j \ne (N+1)/2$$

If the central sample $x_{(N+1)/2}=1$, the output of $f_{B1}(.)$ depends on central sample neighbourhoods, only. To achieve $f_{B1}(.)=1$, besides $x_{(N+1)/2}=1$ the necessary and sufficient condition is that at least $(N-w_{(N+1)/2})/2$ neighbouring samples are equal to 1. Then, there exists the logical product that results in 1 and thus, $f_{B1}(.)$ is equal to 1, too. The above analysis can be summarised as

$$f_{B1}(.)=\begin{cases}1 & \text{if } x_{(N+1)/2}=1 \text{ and } \sum \mathrm{X} \geq (N-w_{(N+1)/2})/2\\ 0 & \text{if } x_{(N+1)/2}=0 \text{ or } \sum \mathrm{X} < (N-w_{(N+1)/2})/2\end{cases} \quad (11)$$

where X represents the set of $N-1$ neighbouring binary samples.

In the case of $f_{B2}(.)$, i.e. logical summing of logical products having $(N+w_{(N+1)/2})/2$ neighbouring samples, the analysis of this Boolean function depends only on the configuration of zeros and ones in the set of neighbouring samples X, and thus, it is more simply than in the case of $f_{B1}(.)$. The arbitrary logical product of $f_{B2}(.)$ is expressed as

$$\underset{i_j \in \Im}{\mathrm{MIN}}\{x_{i_1}, x_{i_2}, \ldots, x_{i_{(N+w_{(N+1)/2})/2}}\} \quad (12)$$

where constraint $\Im$ is defined by

$$1 \leq i_1 \leq i_2 \leq \ldots \leq i_j \leq \ldots \leq i_{(N+w_{(N+1)/2})/2} \leq N$$

$$i_j \neq (N+1)/2$$

The necessary and sufficient condition for $f_{B2}(.)=1$ is that at least $(N+w_{(N+1)/2})/2$ neighbouring samples are equal to 1. Then, at least one logical product of $f_{B2}(.)$ results in 1. This condition can be easily interpreted as

$$f_{B2}(.)=\begin{cases}1 & \text{if } \sum \mathrm{X} \geq (N+w_{(N+1)/2})/2\\ 0 & \text{if } \sum \mathrm{X} < (N+w_{(N+1)/2})/2\end{cases} \quad (13)$$

Finally, the binary output of CWM with the length N and the central weight $w_{(N+1)/2}$ is given by combination of equations (9), (11) and (13) that result in following expression

$$f_{PBF}(.)=\begin{cases}1 & \text{if } x_{(N+1)/2}=1 \text{ and } \sum \mathrm{X} \geq (N-w_{(N+1)/2})/2\\ 1 & \text{if } \sum \mathrm{X} \geq (N+w_{(N+1)/2})/2\\ 0 & \text{otherwise}\end{cases} \quad (14)$$

5. CONCLUSION

The implementation of median-type filters still represents actual and interesting problem, since these filters are investigated for their robustness against outliers and impulse noise. The reason that CWMs can be expressed through PBFs leads to the same filtering operation in the binary domain. In order to simplify the binary outputting of CWMs, the binary analysis based on the monitoring of the configuration of zeros and ones in the binary input set has been provided. Concerning the derived expressions and the proposed analysis, this paper represents a valuable framework for a general design of binary CWMs.

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BIOGRAPHY

Rastislav Lukáč (Ing., Ph.D.) received the M.Sc. (Ing.) degree with honours at the Technical University of Košice, the Slovak Republic, at the Department of Electronics and Multimedia Communications in 1998. In 2001 he finished Ph.D. study. The title of his dissertation work was „New structures of LUM smoothers and impulse detectors for noisy images“, in which he focused on the impulse noise suppression in monochromatic, color and multidimensional images. Currently, he is an assistant professor at the Department of Electronics and Multimedia Communications at the Technical University of Košice.

He is a co-author of the book „Digital filtering of image signal“ and some textbooks focused on electronics, multimedia systems and telecommunications. He is an author of over 60 scientific papers published in journals and conference proceedings. He was an organiser of the international scientific conference „Digital Signal Processing and Multimedia Communications (DSP-MCOM) 2001“.

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